On Sequences of Bounded Variation

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Abstract

In this article, the sequence of bounded variation was considered and its various properties, with examples and counterexamples, were studied in detail. Also, the study focused on the relationship of the sequence of bounded variation with monotonic sequences, convergent and divergent sequences. Further, some necessary and sufficient conditions and sometimes only necessary condition was studied, where sufficient condition is not always true, with examples. The space of the sequence of bounded variation also considered, which is denoted bybv. Here it is shown that bv is closed with respect to addition and multiplication. Hence with respect to some norm bv is a Banach space. Many studies are being found in history regarding the summability of sequence of bounded variation with respect to different types of infinite matrices. Here the summability properties of the sequence of bounded variation were not considered.

Keywords: sequence of bv ; boundedness; convergence; monotonicity

1. Introduction

A few discussions of the sequence of bounded variation are available in Iyer (1977) and Loya, (2006), the nature of the transformed sequence of a sequence of bounded variation by some regular infinite matrix has been studied by various authors. It is also known from the literature of Iyer (1977) and Lahiri (1996) that the space of the sequences of bounded variation is a Banach space with a certain norm. Further some applications of sequences of bounded variation are available in the literature (Kirişci, 2014 & 2016). Hence we shall study a series of results on such sequences in regard to convergence. First, we shall recall and introduce some definitions and results, useful for our study.

DEFINITION-1.1:

Let $x = \{x_n\}$ be any sequence of real numbers and we write $\Delta x_n = x_n - x_{n+1}$, for all$n \in \mathbb{N}$. Then x is said to be of bounded variation (or x is variationally bounded) if $\sum_{n=1}^{\infty} |\Delta x_n|$ is convergent i.e. $\sum_{n=1}^{\infty} \Delta x_n$ is absolutely convergent. Let us denote by bv , the family of all such sequences of real numbers. We write v_n(x)= $\sum_{i=1}^{n} |\Delta x_i|$, for all $n \in \mathbb{N}$. Clearly, $x \in bv$ if $\{v_n(x)\}$ is convergent i.e. if ∃ a real number $k > 0$ such that $v_n(x) < k$ for all $n \in \mathbb{N}$. For simplicity, we write, if $x \in bv$ then x is bv.

DEFINITION-1.2:

Let x =$\{x_n\}_n \in \mathbb{R}^{\mathbb{N}}$. Then the number $v(x)$= $\underset{n \geq 1}{sup} \{v_n(x)\}$, if it exists, is called the total variation of x.

Clearly,$v(x)$≥0 and $v(x)$=$\begin{cases} = 0 & when\ x\ is\ constant\ sequence \\ < \infty & for\ x \in bv \end{cases}$

Thus $v(x)$ may be finite or infinite and $v: \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^+$ where $\mathbb{R}^+$ is the set of non-negative real numbers.

RESULT-1.3: Every absolutely convergent series is convergent but the converse is not necessarily true.

2. MAIN RESULTS

In this section, we shall prove our main theorems. Let c, m , ms(i), ms(d) and D denote respectively the family of convergent sequences, bounded sequences, monotonic increasing sequences, monotonic decreasing sequences and divergent sequences of real numbers.

THEOREM-2.1:

If $x \in bv$ then $x \in c$ but the converse is not always true.

Proof: Let x = $\{x_n\} \in bv \Rightarrow s$ = $\sum_{n=1}^{\infty} \Delta x_n$ is absolutely convergent ⇒ (by result 1.3) s is convergent.

Now, for some $k \in \mathbb{N}$, $s_k = \sum_{i=1}^{k} \Delta x_i$, (the k^{th} partial sum of s) = $x_1 - x_{k+1}$.
s is convergent ⇒ $\lim_{k\to\infty} s_k = l$ ($\in \mathbb{R}$)(say) exists finitely in $\mathbb{R}$.
⇒ $\lim_{k\to\infty} (x_1 - x_{k+1}) = a$, for some real number a ⇒ $\lim_{k\to\infty} x_k = x_1 - a$, which is finite ⇒ $x \in c$.
The converse follows from the following.

Example-2.1:

Let $x = \{x_n\}$ where $x_n = (-1)^n.\frac{1}{n}$, for all n , n=1,2,3,........ Then $\lim_{n\to\infty} x_n = 0$

Again, $|\Delta x_n| = |x_n - x_{n+1}| = \left|(-1)^n.\frac{1}{n} - (-1)^{n+1}.\frac{1}{n+1}\right| = \left|(-1)^n.\{\frac{1}{n} + \frac{1}{n+1}\}\right| = \left|\frac{1}{n} + \frac{1}{n+1}\right|$

$= \frac{2n+1}{n(n+1)} > \frac{n+1}{n(n+1)} = \frac{1}{n}$ (since $2n + 1 > n + 1$) ⇒ $\sum_{n=1}^{\infty} |\Delta x_n| > \sum_{n=1}^{\infty} \frac{1}{n}$ ⇒ $\sum_{n=1}^{\infty} |\Delta x_n|$ is divergent as $\sum_{n=1}^{\infty} \frac{1}{n}$ is so ⇒ $x \notin bv$.

Whereas $x \in c$ ($x \in c$ but x is not monotonic).

COROLLARY-2.2:

If $x \in bv$ then x is cauchy sequence.

Proof: The proof is trivial, as every convergent sequence of Real numbers is cauchy sequence, hence from THEOREM-2.1, the result follows.
It is immediate that the converse is not necessarily true.

THEOREM-2.3:

Every sequence of bounded variation is bounded but the converse is not necessarily true.

Proof: From THEOREM-2.1, it follows that every sequence of bv is convergent and hence bounded.

The converse follows from the following

EXAMPLE-2.2:

Let us consider the sequence x $=\{x_n\}$, where x_n=0, when n is even and $\frac{1}{3}$ when n is odd (i.e. x_{2n}= 0 and x_{2n-1}= $\frac{1}{3}$) ; then clearly x is bounded.

THEOREM-2.4:

Any monotonic sequence is not always bv.

Proof: The proof follows from the following

EXAMPLE-2.3:

Let $x = \{x_n\}$ where $x_n = n$, for all n , n =1,2,3,......... Clearly, x is monotonic increasing i.e.

$x \in ms(i)$. Further $\sum_{n=1}^{\infty} |\Delta x_n| = \sum_{n=1}^{\infty} |n-(n-1)| = \sum_{n=1}^{\infty} 1$ which is divergent

$$\Rightarrow x \notin bv.$$

A similar conclusion can be had for any sequence i.e. $ms(d)$.

Following from THEOREM-2.1 and THEOREM-2.4, a natural question may be posed- under condition(s), a convergent sequence and a monotonic sequence are bv. The following theorem answers both the questions.

THEOREM-2.5:

Every monotonic convergent sequence is bv.

Proof: Let us first consider a monotonic non-decreasing sequence $x = \{x_n\}$ which is convergent to some $l \in \mathbb{R}$. Then $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, n = 1,2,3,......... and $x_n \leq l$ for all n ,

$$n=1,2,3,\ldots \text{Now,} \sum_{n=1}^{\infty} |\Delta x_n| = \sum_{n=1}^{\infty} |x_n - x_{n+1}| = \sum_{n=1}^{\infty} (x_{n+1} - x_n) = \lim_i \sum_{n=1}^{i} (x_{n+1} - x_n) = \lim_i (x_{i+1} - x_1) = l - x_1 < +\infty$$

$$\Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| \text{ is convergent} \Rightarrow x \in bv$$

The conclusion is same for a non-increasing sequence.

NOTE-2.5.1:

Monotonic divergent sequences may not always bv, which follows from the following

EXAMPLE-2.4:

Let $x = \{x_n\}$, where $x_n = n + \frac{1}{n}$, for all $n \in \mathbb{N}$.

Now $x_{n+1} - x_n = n + 1 + \frac{1}{n+1} - n - \frac{1}{n} = 1 + \frac{1}{n+1} - \frac{1}{n} = 1 - \frac{1}{n(n+1)} > 0$ for all $n \in \mathbb{N}$

$\Rightarrow x_n < x_{n+1}$ for all $n \in \mathbb{N} \Rightarrow x$ is strictly increasing sequence and is divergent.

Again, $\sum_{n=1}^{\infty} |\Delta x_n| = \sum_{n=1}^{\infty} |x_n - x_{n+1}| = \sum_{n=1}^{\infty} \left|1 - \frac{1}{n(n+1)}\right| = \sum_{n=1}^{\infty} \{1 - \frac{1}{n(n+1)}\}$

$\geq \sum_{n=1}^{\infty} \frac{n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{(n+1)}$ which is divergent.

$\Rightarrow x \notin bv$.

Similarly, monotonic decreasing divergent sequence is not bv ($x = \{x_n\}$ where $x_n = -n$ will give the conclusion).

THEOREM-2.6:

If $x, y \in bv$, then $x \pm y$, $xy \in bv$ where $x = \{x_n\}$, $y = \{y_n\}$, $x \pm y = \{x_n \pm y_n\}$ and $x\,y = \{x_n y_n\}$.

Proof: $x, y \in bv \Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| < +\infty$ and $\sum_{n=1}^{\infty} |\Delta y_n| < +\infty$ ....(i)
Also, by Theorem-2.3, ∃ M_1, $M_2 > 0$ such that $|x_n| \leq M_1$ and $|y_n| \leq M_2$ for all $n \in \mathbb{N}$.

Let $h = \{h_n\}$ where $h_n = x_n \pm y_n$ for all $n \in \mathbb{N}$.

Then $|\Delta h_n| = |h_n - h_{n+1}| = |(x_n \pm y_n) - (x_{n+1} \pm y_{n+1})| = |(x_n - x_{n+1}) \pm (y_n - y_{n+1})|$

$\leq |x_n - x_{n+1}| \pm |y_n - y_{n+1}| = |\Delta x_n| \pm |\Delta y_n|$

$\Rightarrow \sum_{n=1}^{\infty} |\Delta h_n| \leq \sum_{n=1}^{\infty} |\Delta x_n| + \sum_{n=1}^{\infty} |\Delta y_n| < \infty$ (from (i))

$\Rightarrow h \in bv$ i.e. $x \pm y \in bv$.

Next, let $g = \{g_n\}$ where $g_n = x_n y_n$, for all $n \in \mathbb{N}$.

$$\text{Then, } |\Delta g_n| = |g_n - g_{n+1}| = |x_n y_n - x_{n+1} y_{n+1}| = |(x_n - x_{n+1}) y_n + x_{n+1}(y_n - y_{n+1})| \leq |x_n - x_{n+1}|\,|y_n| + |x_{n+1}|\,|y_n - y_{n+1}| \leq M_1\,|x_n - x_{n+1}| + M_2\,|y_n - y_{n+1}|$$

$$\leq M_1 \sum_{n=1}^{\infty} |\Delta x_n| + M_2 \sum_{n=1}^{\infty} |\Delta y_n| < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} |\Delta g_n| \text{ is convergent} \Rightarrow g \in bv \text{ i.e. } xy \in bv\,.$$

Reciprocal of a sequence of bounded variation may not be of bounded variation; e.g. if for any sequence $\{x_n\}_n$ with $x_n \to 0$ as $n \to \infty$, or x_n is eventually zero, then $\{\frac{1}{x_n}\}$ is not bounded; therefore, to extend this result, we must exclude these types of sequences which are arbitrarily closed to zero.

THEOREM-2.7:

If $x = \{x_n\} \in bv$ and it is bounded away from zero, then $\{\frac{1}{x_n}\} \in bv$.

Proof:

$$x = \{x_n\} \in bv \Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| < +\infty \text{....(i)}$$

$$x \text{ is bounded away from zero}$$

$$\Rightarrow \exists\, M > 0 \text{ such that } |x_n| \geq M, \text{ for all } n \in \mathbb{N} \text{..............(ii)}$$

$$\text{Let } h_n = \frac{1}{x_n}\,, \text{ for all } n \in \mathbb{N}.$$

$$\text{Then } |\Delta h_n| = |h_n - h_{n+1}| = \left|\frac{1}{x_n} - \frac{1}{x_{n+1}}\right| = \frac{|x_{n+1} - x_n|}{|x_n|\,|x_{n+1}|} \leq \frac{1}{M^2}\,|x_{n+1} - x_n| \quad \text{for all } n \in \mathbb{N}$$

$$= \frac{1}{M^2}\,|\Delta x_n|$$

$$\Rightarrow \sum_{n=1}^{\infty} |\Delta h_n| \leq \frac{1}{M^2} \sum_{n=1}^{\infty} |\Delta x_n| < +\infty \Rightarrow \{h_n\} \text{ i.e. } \{\frac{1}{x_n}\} \in bv\,.$$

NOTE-2.7.1:

From Theorem-2.6, the following results are immediate:

$$\text{i) } \sum_{n=1}^{\infty} |\Delta(x_n \pm y_n)| \leq \sum_{n=1}^{\infty} |\Delta x_n| + \sum_{n=1}^{\infty} |\Delta y_n| \leq v(x) + v(y)$$

$$\Rightarrow v(x+y) \leq v(x) + v(y)\,.$$

$$\text{ii) } \sum_{n=1}^{\infty} |\Delta(x_n y_n)| \leq M_2 \sum_{n=1}^{\infty} |\Delta x_n| + M_1 \sum_{n=1}^{\infty} |\Delta y_n| \leq M_2\, v(x) + M_1\, v(y)$$

$$\Rightarrow v(xy) \leq M_2\, v(x) + M_1\, v(y)$$

From Theorem-2.7, it follows that

$$\text{iii) } \sum_{n=1}^{\infty} \Delta\left(\frac{1}{x_n}\right) \leq \frac{1}{M^2} \sum_{n=1}^{\infty} |\Delta x_n| \leq \frac{1}{M^2} v(x) \Rightarrow v\left(\frac{1}{x}\right) \leq \frac{1}{M^2} v(x).$$

Let $x \in bv$ and $\lambda \in \mathbb{R}$. Then $v(\lambda x) \leq |\lambda| v(x)$ ....

In fact $x \in bv$ and $\lambda \in \mathbb{R} \Rightarrow \sum_{n=1}^{\infty} \Delta(\lambda x_n) = \sum_{n=1}^{\infty} |\lambda x_n - \lambda x_{n+1}| = \sum_{n=1}^{\infty} |\lambda| |x_n - x_{n+1}|$

$$= |\lambda| \sum_{n=1}^{\infty} |\Delta x_n| \leq |\lambda| v(x) \Rightarrow v(\lambda x) \leq |\lambda| v(x).$$

THEOREM-2.8:

Any sequence $x = \{x_n\}$ is of bv iff it can be expressed as a difference of two monotonic convergent sequences i.e. there exists monotonic convergent sequences $\{u_n\}$ and $\{v_n\}$ such that $x_n = u_n - v_n$, for all n , n =1,2,3,.........

Proof: Necessary Part:

Let $x = \{x_n\} \in bv \Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| < +\infty$.

We write $u_n = \sum_{i=n}^{\infty} |\Delta x_i|$, for all $n \in \mathbb{N}$ .Then, clearly $\{u_n\}$ is a monotonic non-increasing and bounded , as $\sum_{n=1}^{\infty} |\Delta x_n|$ is converged (so is $\{u_n\}$).

We define $v_n = u_n - x_n$, for all $n \in \mathbb{N}$. Then $\{v_n\}$ is convergent as both of $\{u_n\}$ and $\{x_n\}$ are convergent (following from Theorem-2.1).

Now, $v_n - v_{n+1} = u_n - x_n - u_{n+1} + x_{n+1} = (u_n - u_{n+1}) - (x_n - x_{n+1})$

$$= \sum_{i=n}^{\infty} |\Delta x_i| - \sum_{i=n+1}^{\infty} |\Delta x_i| - (x_n - x_{n+1})$$

$$= |\Delta x_n| + \sum_{i=n+1}^{\infty} |\Delta x_i| - \sum_{i=n+1}^{\infty} |\Delta x_i| - (x_n - x_{n+1}) = |\Delta x_n| - \Delta x_n \geq 0.$$

Which is true for all $n \in \mathbb{N}$. This shows that $\{v_n\}$ is monotonic non-increasing. Hence, by theorem-2.5, $\{v_n\} \in bv$.

Thus, $x_n = u_n - v_n$, for all $n \in \mathbb{N}$, where $\{u_n\}$, $\{v_n\} \in bv$, whenever $\{x_n\} \in bv$. Also, $\{u_n\}$ and $\{v_n\}$ are monotonic non-increasing sequences which are convergent. Similar conclusion can be made using monotonic non-decreasing convergent sequences.

Hence every sequence of bv can be expressed as the difference of two monotonic convergent sequences.

Sufficient Part:

Let $\{x_n\}$ be any sequence which can be expressed as the difference of two monotonic convergent sequences, say, $\{u_n\}$ and $\{v_n\}$ i.e. $x_n = u_n - v_n$, for all $n \in \mathbb{N}$.

Now $\{u_n\}$ and $\{v_n\}$ are monotonically convergent

$\Rightarrow \{u_n\}$ and $\{v_n\}$ are variationally bounded (by Theorem-2.5)

$\Rightarrow \{u_n - v_n\}$ is variationally bounded (by Theorem-2.6)

$\Rightarrow \{x_n\}$ is variationally bounded .

Hence the theorem is proved.

REMARK-2.1:

1. Any divergent (also non-convergent) sequence may not be of bounded variation.

Proof: Let us consider the sequence $x = \{x_n\}$ where $x_n = +\frac{1}{n}$, for all $n \in \mathbb{N}$. Clearly, x is divergent.

$$\text{Again, } \Delta x_n = x_n - x_{n+1} = (n + \frac{1}{n}) - (n + 1 + \frac{1}{n+1}) = \frac{1}{n(n+1)} - 1 < 0$$

$$[n + 1 > n \Rightarrow n(n + 1) > n^2 \geq 1 \text{ ,as } n \geq 1 \Rightarrow \frac{1}{n(n+1)} < 1 \text{ for all } n \in \mathbb{N}]$$

$$\Rightarrow x \text{ is strictly decreasing.}$$

$$\text{Also, } \sum_{n=1}^{\infty} |\Delta x_n| = \sum_{n=1}^{\infty} \left| \frac{1}{n(n+1)} - 1 \right| = \sum_{n=1}^{\infty} (1 - \frac{1}{n(n+1)}) = \sum_{n=1}^{\infty} \frac{n^2+n-1}{n(n+1)} \geq \sum_{n=1}^{\infty} \frac{n}{n(n+1)}$$

$$[n^2 \geq 1 \Rightarrow n^2 + n \geq n + 1 \Rightarrow n^2 + n - 1 \geq n]$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)} \text{ which is divergent.}$$

$$\Rightarrow x \text{ is not of bounded variation.}$$

Further, we assume the sequence $x = \{x_n\}$ where $x_n = (-1)^n$, for all $n \in \mathbb{N}$. Clearly, x is bounded and non-convergent.

$$\text{Also, } |\Delta x_n| = |x_n - x_{n+1}| = |(-1)^n - (-1)^{n+1}| = |(-1)^n - (-1)^n (-1)|$$

$$= |(-1)^n + (-1)^n| = |2(-1)^n| = 2. |(-1)^n| = 2. \qquad [\text{since } |(-1)^n| = 1].$$

$$\Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| = \sum_{n=1}^{\infty} 2 \text{ which is clearly divergent} \Rightarrow x \text{ is not bounded variation.}$$

Hence it's the result.

2. The difference of a monotonic convergent sequence and a monotonic divergent sequence may not be of bounded variation.

Proof: Let us consider two sequences $u=\{u_n\}$ and $v=\{v_n\}$ where $u_n = -n$ and $v_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then clearly, u is monotonic divergent (negative) and v is a monotonic convergent (positive) sequence.

We write $x_n = u_n - v_n = -n - \frac{1}{n} = -\frac{n^2+1}{n}$ for all $n \in \mathbb{N}$.

$$\text{Now, } |\Delta x_n| = |x_n - x_{n+1}| = \left|-\frac{n^2+1}{n} + \frac{(n+1)^2+1}{n+1}\right| = \left|n+1+\frac{1}{n+1} - n - \frac{1}{n}\right| = \left|1 - \frac{1}{n(n+1)}\right|$$

$$= \left|\frac{n^2+n-1}{n(n+1)}\right| \geq \left|\frac{n}{n(n+1)}\right| = \frac{1}{n+1} \qquad [\text{as } n^2+n-1 \geq n]$$

$$\Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| \geq \sum_{n=1}^{\infty} \frac{1}{n+1} \Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| \text{ is divergent as R.H.S. is divergent.}$$

This shows that $\{x_n\}$ is not of bounded variation.

3. The difference of two monotonic divergent sequences may not be of bounded variation.

Proof:

Let $u_n = n + \frac{(-1)^n}{n}$, for all $n \in \mathbb{N}$ and $v_n = n$, for all $n \in \mathbb{N}$. Then, both $\{u_n\}$ and $\{v_n\}$ are monotonic divergent. Let's write $x_n = u_n - v_n = \frac{(-1)^n}{n}$, for all $n \in \mathbb{N}$.

$$\text{Now, } |\Delta x_n| = |x_n - x_{n+1}| = \left|\frac{(-1)^n}{n} - \frac{(-1)^{n+1}}{n+1}\right| = \left|\frac{(-1)^n}{n} + \frac{(-1)^n}{n}\right| = \frac{n+1+1}{n(n+1)} = \frac{2n+1}{n(n+1)} > \frac{n+1}{n(n+1)}$$

$$= \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| > \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} |\Delta x_n| \text{ is divergent as } \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent} \Rightarrow$$

$\{x_n\}$ is not of bounded variation.

Now the behavior of the family bv of sequences of bounded variation is studied below:

THEOREM-2.9: (Lahiri, 1996)

The family bv of sequences of bounded variation forms a vector space over the field $\mathbb{R}$ of real numbers under the addition and scalar multiplication defined in it by:
for all $x, y \in bv$, $\lambda \in \mathbb{R}$, $x + y = \{x_n + y_n\}$ and $\lambda x = \{\lambda x_n\}$ where $x = \{x_n\}$ and $y = \{y_n\}$.

Further, the space bv will be a Banach Space under the norm defined by [2.1]

$$\|x\| = |x_1| + \sum_{n=1}^{\infty} |x_n - x_{n+1}|, \text{ for all } x \in bv .$$

Proof is easy.

3. Conclusion

The study focused on some properties of the sequence of bv and also investigated the relationship of the sequence of bv with bounded, convergent, Cauchy and monotonic sequences, with sufficient examples and counterexamples for converse. Also, it is found that monotonic convergent sequence is bv but not a monotonic divergent sequence. Further, the algebra of sequences of bv was studied, and noted that the ratio of two sequences of bv may not be so; it is bv if the denominator sequence is bounded away from 0. It is also found that a necessary and sufficient condition for a sequence of bv can be expressed as a difference of two monotonic convergent sequences.

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